Math 131B-2: Homework 7

Due: May 19, 2014

1. Read Apostol Sections 9.12, 9.18-19, 8.24, 9.22.

2. Find the intervals of convergence of the following power series:

   - $\sum_{n=1}^{\infty} \frac{2^n}{n^2}(x-5)^n$
   - $\sum_{n=1}^{\infty} \sqrt{n}(x-1)^n$
   - $\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}(x+2)^n$
   - $\sum_{n=1}^{\infty} n!(x-3)^n$

3. Prove that the power series $f(x) = \sum_{n=0}^{\infty} x^n$ is not uniformly convergent on $(-1,1)$.

4. Do Apostol problems 9.33, 9.36, and 9.37.

5. Do Apostol problem 9.21. Explain why the result you get does not contradict Abel's Theorem. Note: for this problem, it will be helpful to recall the partial fractions decomposition $\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)$.

6. Recall that $B(M \to S)$ is the set of bounded functions $f\colon M \to S$ and $C(M \to S) \subset B(M \to S)$ is the subspace consisting of bounded continuous functions.

   - Prove that the operation $d_\infty(f,g) = \sup_{x\in M} d_S(f(x),g(x))$ is a metric on $B(M \to S)$.
   - We showed in class that $C(M \to S)$ is a closed subset of $B(M \to S)$. Give an example showing that $C(M \to S)$ need not be compact.
   - Give an example showing that if S is not complete, $C(M \to S)$ need not be complete.

7. *Double sums with only nonnegative terms*

   - Prove that $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$ if $a_{ij} \geq 0$ for all i and j, as long as we allow the case $+\infty = +\infty$ to occur.
   - Let
   $$a_{ij} = \frac{1}{(i+1)!}\left(\frac{i}{i+1}\right)^j.$$
   What is $\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ij}$?

8. *Power series and integral approximations* We can use what we know about integrating power series to approximate integrals we can't do by hand. Here's an example in which we approximate $\int_0^1 \cos(x^2)$.

   - Prove that $\int_0^1 \cos(x^2) = \sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)(2n)!}$.
   - Recall that if $s = \sum_{n=0}^{\infty}(-1)^n a_n$ is an alternating series and s_n is its nth partial sum, we have $|s_n - s| < a_{n+1}$. With this in mind, find an approximation to $\int_0^1 \cos(x^2)$ which is within 10^{-3} of the true value. Is your answer an over or underestimate?